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On the universal norm distribution

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Abstract. We introduce and study the universal norm distribution in this paper, which generalizes the concepts of universal ordinary distribution and the universal Euler system. We study the Anderson type resolution of the universal norm distribution and then use this resolution to study the group cohomology of the universal norm distribution.

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1. Introduction

Let r be a positive integer, the *universal ordinary distribution* of rank 1 and level r is well known to be the free abelian group

$$U_r = \frac{\langle [a] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}\rangle}{\langle [a] - \sum_{pb=a}[b] : p \mid r, a \in \frac{p}{r}\mathbb{Z}/\mathbb{Z}\rangle}.$$

With a natural $G_r = \mathrm{Gal}(\mathbb{Q}(\mu_r)/\mathbb{Q})$ action on U_r, U_r becomes a G_r-module and plays a very important role in the study of cyclotomic fields, see for example Lang [4] or Washington [10] for more information. In particular, the sign cohomology of U_r gives key information about the indices of cyclotomic units and Stickelberger ideals as illustrated by Sinnott's original paper [9] and many following papers on this subject by different authors. The G_r-cohomology is found to be related to the cyclotomic Euler system, as shown by Anderson-Ouyang [1] about the Kolyvagin recursion in the universal ordinary distribution.

In the book [8], Rubin defined a generalization of the universal ordinary distribution, which he called the *universal Euler system*. It then was used to prove the Kolyvagin recursions satisfied by the Euler systems. However, there are other universal objects satisfying similar distribution relations. In the paper [6], we proposed a generalization of the universal ordinary distribution, have the following canonical decomposition

$$G_z = \prod_{x|z} G_{z(x)}.$$

Let N_z be the sum of all elements $g \in G_z$ in the group ring $\mathbb{Z}[G_z]$. For z finite and $z' \mid z$, Let $g_{z'}$ denote the image of $g \in G_z$ in $G_{z'}$. Let $N_{z'}^{z}$ be the corresponding inflation map from $\mathbb{Z}[G_{z'}]$ to $\mathbb{Z}[G_z]$. For every infinite $\mathbf{z} \in Z$, let $G_{\mathbf{z}}$ be the inverse limit of G_z over all finite $z \mid_s \mathbf{z}$. Then $G_{\mathbf{z}}$ is actually the direct product of $G_{\mathbf{z}(x)}$ for every $x \mid \mathbf{z}$.

Write $B_z = \{[gz] : g \in G_z\}$, then

$$A = \bigcup_{z \in Z_{\text{fin}}} B_z = \{[gz] : g \in G_z, z \in Z_{\text{fin}}\},$$

and G_{x^n} acts trivially in B_z if $x \nmid z$. Thus A and $\{G_z : z \in Z_{\text{fin}}\}$ are uniquely determined by each other. Let $A_z = \bigcup_{z'|_s z, z' \in Z_{\text{fin}}} B_{z'}$ for every $z \in Z$.

For each pair $x \in X$ and $z \in Z$, the Frobenius element Fr_x is a given element in G whose restriction to G_{x^n} is the identity for every $n \in \mathbb{N}$.

Let $\mathcal{O}$ be an integral domain and let Φ be its fractional field. Let $\mathcal{T}$ be a fixed $\mathcal{O}$-algebra which is torsion free and finitely generated as an $\mathcal{O}$-module. We suppose that $\mathcal{T}$ is a trivial G-module. For each $x \in X$, a polynomial

$$p(x; t) \in \mathcal{T}[\,t\,]$$

is chosen corresponding to x.

2.2 Definition of the universal norm distribution

Let $\mathcal{A}$ be the free $\mathcal{T}$-module generated by A, along with the G-action, $\mathcal{A}$ becomes a torsion free $\mathcal{T}[G]$-module. Let $\mathcal{B}_z$ be the $\mathcal{T}[G]$-submodule of A generated by B_z as $\mathcal{T}$-module for $z \in Z_{\text{fin}}$. Then $\mathcal{B}_z$ is nothing but a free rank 1 $\mathcal{T}[G_z]$-module with generator $[z]$. Let $\mathcal{A}_z$ be the $\mathcal{T}[G]$-submodule generated by A_z as $\mathcal{T}$-module for every $z \in Z$. Thus $\mathcal{A}_z$ has a natural $\mathcal{T}[G_{z'}]$-module structure for every $z \mid_s z'$.

Let $\lambda_{z(x)}$ be the $\mathcal{T}[G_z]$-homomorphism of $\mathcal{A}_z$ given by

$$\lambda_{z(x)} : [z'] \longmapsto \begin{cases} p(x; \text{Fr}_x^{-1})[z'] - N_{z(x)}[z(x)z'], & \text{if } x \nmid z', \\ 0, & \text{if } x \mid z'. \end{cases}$$

Let $\mathcal{D}_z$ be the submodule of $\mathcal{A}_z$ generated by the images of $\lambda_{z(x)}(\mathcal{A}_{z/z(x)})$ for all $x \mid z$. Elements in $\mathcal{D}_z$ are called *distribution relations* in $\mathcal{A}_z$. The *universal norm distribution* $\mathcal{U}_z$ according to the above assumptions is defined to be the quotient $\mathcal{T}[G_z]$-module $\mathcal{A}_z/\mathcal{D}_z$, i.e., $\mathcal{A}_z$ modulo all distribution relations.

of f with respect to the Carlitz module. The Galois group G_f of $K(f)/K$ is known to be isomorphic to $(R/f)^\times$. Thus we can identify every $\sigma = \sigma_x \in G_f$ for some(a unique) $x \in (R/f)^\times$. The *ordinary distribution of level f* on the function field K is defined to be a map

$$\phi : \frac{1}{f}R/R \longrightarrow Ab = \text{abelian group}$$

satisfying

$$\phi(x) = \sum_{py=x} \phi(y), \forall p \mid f, x \in \frac{p}{f}R/R.$$

One can then talk about the *universal ordinary distribution* as the universal object to the category of ordinary distributions. As in the number theory counterpart, by abusing notation, we say the group

$$U_f = \frac{\langle [a] : a \in \frac{1}{f}R/R\rangle}{\langle [a] - \sum_{pb=a}[b] : p \mid f, a \in \frac{p}{f}R/R\rangle}$$

the universal ordinary distribution. U_f is naturally equipped with a G_f-action by sending $\sigma_x[a] = [xa]$. The distribution U_f, as shown to be a free abelian group of order $|G_f|$, plays a similar role to the universal ordinary distribution in the study of cyclotomic function field,

Now let $G = G_K = \mathrm{Gal}(K^{sep}/K)$. Let X be the set of all monic irreducible polynomials in K and then Z_{fin} is nothing but the set of all monic polynomials in R. Let A be the discrete set $\{[g \circ f] : f \in Z_{\text{fin}}, g \in G_f\}$. Then G acts on A by setting $g \circ [f] = [f]$ if $g \in G_{K(f)}$. Let $p(\wp, t) = 1 - t$ for every $\wp \in X$. For $\mathcal{O} = \mathcal{T} = \mathbb{Z}$, we then can define the universal norm distribution $\mathcal{U}_f$ as the G_f-module

$$\mathcal{U}_f = \frac{\langle [\sigma f'] : f' \mid_s f, \sigma \in G_{f'}\rangle}{\langle (1 - \mathrm{Fr}_p^{-1})[\sigma f'] - N_{f(p)}[\sigma f(p) f'] : f(p) f' \mid_s f, \sigma \in G_{f'}\rangle}.$$

Proposition 3.4. *The modules U_f and $\mathcal{U}_f$ are isomorphic as G_f-modules by identifying $[1/f'] \in U_f$ and $[f'] \in \mathcal{U}_f$.*

Proof. The proof is similar to Proposition 3.1. One can easily check that: (1). the map $[1/f'] \in U_f \mapsto [f'] \in \mathcal{U}_f$ is well defined; (2). this map is a G_f-morphism; (3). surjective; (4). both U_f and $\mathcal{U}_f$ have $\mathbb{Z}$-rank $|G_f|$(the latter follows from Proposition 4.1). □

3.6 Function field case: II

We now work on more generality. Let K be a fixed function field. Pick a place ∞ in K. Let R be the integer ring corresponding to the place ∞. Choose a sign then $\mathcal{L}_z$ becomes a graded $\mathcal{T}[G_z]$-module. $\mathcal{L}_z$ is bounded above since all its non-negative components are 0. Moreover, $\mathcal{L}_z$ is bounded if and only if z is finite.

With abuse of notation, denote by $\lambda_{z(x)}, \lambda_{z'}$ the homomorphisms of $\mathcal{L}_z$ inheriting from the homomorphisms in $\mathcal{A}_z$ bearing the same names. Now let

$$d : \mathcal{L}_z \longrightarrow \mathcal{L}_z,\ [a, y] \longmapsto \sum_{x|y} \omega(x, y)\lambda_{z(x)}[a, y/x]$$

where ω is as defined in § 2.1. Clearly d commutes with G_z-actions. A straightforward calculation shows that $d^2 = 0$ and therefore d is a differential of degree 1. Define an $\mathcal{T}[G_z]$-homomorphism $\mathbf{u} : \mathcal{L}_z \to \mathcal{U}_z$ by

$$[a, y] \longmapsto \begin{cases} [a], & \text{if } y = \mathbf{1}; \\ 0, & \text{if } y \neq \mathbf{1}. \end{cases}$$

Regard $\mathcal{L}_z$ as a complex $\mathcal{L}_z^\bullet$ by the differential d, and regard $\mathcal{U}_z$ as a complex concentrated on 0-component. Then one can easily check that $\mathbf{u}$ is a homomorphism of complexes. Because of the following Theorem, we call the complex $(\mathcal{L}_z^\bullet, d)$(or simply $\mathcal{L}_z^\bullet$) *Anderson's resolution* of the universal norm system $\mathcal{U}_z$.

Theorem 5.1. *The homomorphism* $\mathbf{u}$ *is a quasi-isomorphism, i.e., the complex* $(\mathcal{L}_z^\bullet, d)$ *is acyclic for degree* $n \neq 0$ *and* $H^0(\mathcal{L}_z^\bullet, d) \cong \mathcal{U}_z$ *induced by* $\mathbf{u}$.

Proof. For any $a \in B_0 \cap B_{z/z(y)}$, consider the graded $\mathcal{T}$-submodule $C_a^\bullet$ of $\mathcal{L}_z^\bullet$ generated by

$$\{\lambda_w[a, y'],\ w \mid_s z, \bar{w}y' \mid y\}.$$

One can see that $C_a^\bullet$ is d-stable. Thus $C_a^\bullet$ is actually a subcomplex of $\mathcal{L}_z^\bullet$. By Proposition 4.1, $\mathcal{L}_z^\bullet$ is the direct sum of $C_a^\bullet$ for a over $B_0 \cap A_z$. We hence only have to study the complex $C_a^\bullet$. Now the theorem follows from Lemma 5.2. □

5.2 The Koszul complex $\tilde{C}_y^\bullet$

Let Λ be the polynomial ring

$$\Lambda = \mathcal{T}[Z] = \{\sum t_z z : t_z \in T, z \in Z\}.$$

Let $\tilde{C}_y^\bullet$ be the Koszul complex of Λ with the regular sequence $x_1 < \cdots < x_m$ where $y = x_1 \cdots x_m$. Thus $\tilde{C}_y^\bullet$ is the graded exterior algebra

$$\bigoplus_{y'|y} \Lambda e_{y'}$$

with the symbol $[x^n]$ is of degree n and the differential given by

$$\partial_{z(x)}[x^n] = \begin{cases} (1-\sigma_{z(x)})[x^{n-1}], & \text{if } n > 0 \text{ odd;} \\ N_{z(x)}[x^{n-1}], & \text{if } n > 0 \text{ even.} \end{cases}$$

Now let $P_{z\bullet}$ as the tensor product of $P_{z(x)\bullet}$ over all $x \mid z$. $P_{z\bullet}$ is the so called *tensor projective resolution* of the trivial $\mathbb{Z}[G_z]$-module $\mathbb{Z}$ with respect to the cyclic decomposition

$$G_z = \prod_{x|z} G_{z(x)} = \prod_{x|z} \langle \sigma_{z(x)} \rangle.$$

Let $[w]$ be an indeterminate for every $w \in Z$. Then the tensor resolution $P_{z\bullet}$ is the projective $\mathbb{Z}[G_z]$-resolution of the trivial module $\mathbb{Z}$ by

$$P_{z,n} = \bigoplus_{\substack{\bar{w}|z \\ \deg w = n}} \mathbb{Z}[G_z][w]$$

and the differential ∂_z is given by

$$\partial_z[w] = \sum_{x|w} (-1)^{\sum_{x'<x} v_{x'} w} \alpha_{z(x)}[w/x]$$

where $\alpha_{z(x)}$ is equal to $\sigma_{z(x)} - 1$ if $v_x w$ odd and $N_{z(x)}$ if $v_x w$ even. For any $z' \mid_s z$, one has a natural inclusion of $P_{z'\bullet}$ to $P_{z\bullet}$ by sending $[w]$ to $[w]$.

6.3 G_z-cohomology of trivial module A

Let A be a free $\mathcal{O}$-module with trivial G_z-structure. To compute its G_z-cohomology, it suffices to compute the cohomology

$$I^\bullet_{A,z} = \mathrm{Hom}_{\mathbb{Z}[G_z]}(P_{z\bullet}, A) = \bigoplus_{\substack{w \text{ finite} \\ \bar{w}|z}} A[w]$$

with the differential

$$\delta_z[w] = \sum_{x|z} (-1)^{\sum_{x'<x} v_{x'} w} a_{z(x)}[wx]$$

where $a_{z(x)}$ is equal to 0 if $v_x w$ even and to $|G_{z(x)}|$ if $v_x w$ odd. The inclusion of $P_{z'\bullet}$ to $P_{z\bullet}$ for $z' \mid_s z$ thus induces a projection from $I^\bullet_{A,z}$ to $I^\bullet_{A,z'}$. One see that $I^\bullet_{A,z'}$ is a direct summand of $I^\bullet_{A,z}$.

For any finite w with $\bar{w} \mid z$, let

$$I^\bullet_A[w^2] = \bigoplus_{w'|\bar{w}} A[w^2/w'],$$

We call the basis $\{c(y, w) : y \mid \bar{w} \mid z\}$ given in Theorem 7.8, the *canonical basis* for $H^*(G_z, \mathcal{U}_z/M\mathcal{U}_z)$. In particular, we write $c(y, y)$ as c_y. By the above theorem, we see that for every $z \in Z$,

$$H^0(G_z, \mathcal{U}_z/M\mathcal{U}_z) = \langle c_y : \ y \mid z\rangle_{\mathcal{T}/M\mathcal{T}}$$

is the union of all $H^0(G_{z'}, \mathcal{U}_{z'}/M\mathcal{U}_{z'})$ with $z' \mid_s z$ and z' finite. We'll use this fact in Ouyang [7] for the double complex $(\mathbf{K}_z^{\bullet,\bullet}; \tilde{d}, \tilde{\delta})$.

Remark 7.10. One can expect parallel result to Theorem B in Ouyang [5] holds here too. The answer is yes. However, we feel more appropriate to state it in Ouyang [7], as a natural consequence of the universal Kolyvagin recursion, just like the proof of the above Theorem B in Anderson and Ouyang [1].

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