Conjectures and Questions in the Value Distribution Theory

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Abstract. Everybody admits that conjectures and questions provide impetus to research in Mathematics. The value distribution theory is in no way an exception. In the article, we present some well-known conjectures and questions in the value distribution theory and the research that they have influenced.

1. The Nevanlinna theory

The focal theme of the value distribution theory is to study the behaviour of the roots of $f(z) - a = 0$ and the manner in which those are distributed over the complex plane, where a is a complex number and f is an entire or a meromorphic function. Rolf Nevanlinna developed a systematic study of the value distribution theory by means of his first and second fundamental theorems. Let us now explain some preliminaries of Nevanlinna theory.

Let f be a non-constant meromorphic function in the complex plane $\mathbb{C}$. We denote by $n(r, a; f)$ the number of roots, counted according to multiplicity, of the equation $f(z)-a = 0$ in $|z| \leq r$ for $a \in \mathbb{C} \cup \{\infty\}$, where the roots of $f(z) - \infty = 0$ are taken as the roots of $\frac{1}{f(z)} = 0$. Henceforth the roots of $f(z) - a = 0$ will be called the *a-points of f*. We put

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r,$$

which is called the *integrated counting function* of the a-points of f.

Also we define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

and call it the *proximity function of f*, where $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $0 \leq x \leq 1$.

For $a \in \mathbb{C}$ we set $m(r, a; f) = m\big(r, \frac{1}{f-a}\big)$ and for $a = \infty$ we put $N(r, \infty; f) = N(r, f)$ and $m(r, \infty; f) = m(r, f)$.

The quantity

$$m(r, a; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta$$

measures the mean deviation (in the reciprocal sense or, more geometrically, as viewed from ∞) of the values of f from the value a for z varying over the circle $|z| = r$. To be more precise: we see that when the values of f are relatively far away from the value a for z on $|z| = r$, then $m(r, a; f)$ is small. On the other hand, if the values of f are relatively close to the value a for z on $|z| = r$, then $m(r, a; f)$ is large.

The quantity $N(r, a; f)$ is large or small according as $f(z) - a = 0$ has relatively many or relatively few roots in the closed disc bounded by $|z| = r$.

The function $T(r, f) = m(r, f) + N(r, f)$ is called the *characteristic function of f*. The function $T(r, f)$ is an increasing convex function of $\log r$.

The **first fundamental theorem of Nevanlinna** {[36], see also p. 5 [21]} states that

$$m(r, a; f) + N(r, a; f) = T(r, f) + O(1)$$

for $a \in \mathbb{C} \cup \{\infty\}$, where $O(1)$ denotes a bounded quantity depending only on $a \in \mathbb{C} \cup \{\infty\}$.

From the first fundamental theorem of Nevanlinna it follows that if f has many a-points in the set $\{z \in \mathbb{C} : |z| \leq r\}$, i.e., if $N(r, a; f)$ is large, then $m(r, a; f)$ is comparatively small, i.e., on $|z| = r$ the values of f are relatively far away from a. On the other hand, if on $|z| = r$ the values of f are relatively close to the value a, then f has comparatively fewer a-points in $\{z \in \mathbb{C} : |z| \leq r\}$.

The quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)}$$

is called the *Nevanlinna deficiency of the value a*. Clearly $0 \leq \delta(a; f) \leq 1$ and the larger the value of $\delta(a; f)$ the fewer the number of a-points of f in $\mathbb{C}$.

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We denote by $\bar{N}(r, a; f)$ the integrated counting function of **distinct** a-points of f. The quantity

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, a; f)}{T(r, f)}$$

is called the *ramification index of the value a*. Clearly $0 \le \delta(a; f) \le \Theta(a; f) \le 1$ and the larger the value of $\Theta(a; f)$ the fewer the number of distinct a-points of f.

We usually denote by $S(r, f)$ any quantity, which satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite linear measure.

2. Nevanlinna's conjecture

Let us start with the most celebrated second fundamental theorem of R. Nevanlinna {[36], see also p. 23 [50]}.

Theorem 1. *Let f be a nonconstant meromorphic function in $\mathbb{C}$. Suppose that $a_1, a_2, \ldots, a_q$ $(q \ge 3)$ be distinct finite complex numbers. Then*

$$(q-2)T(r, f) \le \sum_{\nu=1}^{q} \bar{N}(r, a_\nu; f) + S(r, f).$$

A meromorphic function $a = a(z)$ is called a *small function of a meromorphic function f* if $T(r, a) = S(r, f)$.

From the definition one may anticipate that a small function of f has reasonably slower growth in comparison to that of f. On the other hand, a constant has no growth at all. So it is a most natural curiosity to see the position of Nevanlinna's second fundamental theorem when the distinct constants $a_1, a_2, \ldots, a_q$ are replaced by distincts small functions $a_1(z), a_2(z), \ldots, a_q(z)$. The following conjecture of R. Nevanlinna [36] well addresses this context:

Nevanlinna's Conjecture. *The second fundamental theorem is also valid for small functions as targets.*

Nevanlinna himself settled the conjecture for three small functions as targets {[36]; see also p. 47, Theorem 2.5 [21]}.

Theorem 2. *Let f be a nonconstant meromorphic function and a_1, a_2, a_3 be three distinct small functions of f. Then*

$$T(r, f) \le \sum_{\nu=1}^{3} \bar{N}(r, 0; f - a_\nu) + S(r, f).$$

Proof. We put $\phi(z) = \frac{(f-a_1)(a_2-a_3)}{(f-a_3)(a_2-a_1)}$. By the second fundamental theorem we get

$$T(r, \phi) \le \bar{N}(r, \infty; \phi) + \bar{N}(r, 0; \phi) + \bar{N}(r, 1; \phi) + S(r, \phi). \tag{1}$$

Now

$$\begin{aligned} T(r, f) &\le T(r, f - a_3) + T(r, a_3) + O(1) \\ &= T\left(r, \frac{1}{f - a_3}\right) + S(r, f) \\ &\le T\left(r, \frac{a_3 - a_1}{f - a_3}\right) + S(r, f) \\ &\le T\left(r, 1 + \frac{a_3 - a_1}{f - a_3}\right) + S(r, f) \\ &= T\left(r, \frac{f - a_1}{f - a_3}\right) + S(r, f). \end{aligned}$$

Since $T\left(r, \frac{a_2-a_3}{a_2-a_1}\right) = S(r, f)$, we get

$$\begin{aligned} T(r, f) &\le T\left(r, \frac{f - a_1}{f - a_3}\right) + S(r, f) \\ &= T\left(r, \phi\frac{a_2 - a_1}{a_2 - a_3}\right) + S(r, f) \\ &\le T(r, \phi) + T\left(r, \frac{a_2 - a_1}{a_2 - a_3}\right) + S(r, f) \\ &= T(r, \phi) + S(r, f). \end{aligned} \tag{2}$$

Also

$$\begin{aligned} T(r, \phi) &= T\left(r, \frac{(f - a_1)(a_2 - a_3)}{(f - a_3)(a_2 - a_1)}\right) \\ &\le T\left(r, \frac{f - a_1}{f - a_3}\right) + S(r, f) \\ &= T\left(r, 1 + \frac{a_3 - a_1}{f - a_3}\right) + S(r, f) \\ &\le T\left(r, \frac{1}{f - a_3}\right) + S(r, f) \\ &= T(r, f - a_3) + S(r, f) \\ &\le T(r, f) + S(r, f). \end{aligned} \tag{3}$$

Combining (2) and (3) we get

$$T(r, \phi) = T(r, f) + S(r, f). \tag{4}$$

Hence $S(r, \phi)$ is replaceable by $S(r, f)$.

Finally the equations $\phi(z) = 0, 1, \infty$ have roots only if either $f(z) - a_\nu(z) = 0$ for $\nu = 1, 2, 3$ or if two of the functions a_ν's become equal. Thus

$$\begin{aligned}
&\bar{N}(r, \infty; \phi) + \bar{N}(r, 0; \phi) + \bar{N}(r, 1; \phi) \\
&\quad \le \sum_{\nu=1}^{3} \bar{N}(r, 0; f - a_\nu) + \bar{N}(r, 0; a_1 - a_2) \\
&\quad\quad + \bar{N}(r, 0; a_2 - a_3) + \bar{N}(r, 0; a_3 - a_1) \\
&\quad = \sum_{\nu=1}^{3} \bar{N}(r, 0; f - a_\nu) + S(r, f). \qquad (5)
\end{aligned}$$

Now by (1), (4) and (5) we get $T(r, f) \le \sum_{\nu=1}^{3} \bar{N}(r, 0; f - a_\nu) + S(r, f)$. □

The general case remained open till 2004. Let us now present a brief description of the research done on "Nevanlinna's Conjecture".

After Nevanlinna it is C. T. Chuang [8] who worked on the conjecture in 1964. We state the result of Chuang as follows:

Theorem 3. *Let f be a nonconstant meromorphic function. Let $\psi_l (l = 1, 2, \ldots, p; p \ge 1)$ be p linearly distinct meromorphic functions satisfying $T(r, \psi_l) = o\{T(r, f)\}$ as $r \to \infty$, $l = 1, 2, \ldots, p$, and their q linearly distinct combinations with constant coefficients be*

$$\phi_j = \sum_{k=1}^{p} C_{jk} \psi_k, \quad j = 1, 2, \ldots, q; \quad q \ge 2.$$

Then

$$\begin{aligned}
(q - 1)T(r, f) \le \sum_{j=1}^{q} N_p(r, 0; f - \phi_j) \\
+ p\bar{N}(r, \infty; f) + S(r, f),
\end{aligned}$$

where $S(r, f) = O\{\log T(r, f) + \log r\}$ as $r \to \infty$ through all values if f is of finite order and outside a set of finite linear measure otherwise, and $N_p(r, 0; f - \phi_j)$ denotes the integrated counting function of the roots of $f - \phi_j = 0$, where a root of multiplicity m being counted m times if $m \le p$ and p times if $m > p$.

Following theorem readily follows from the above result of Chuang.

Theorem 4. *Suppose that f is a meromorphic function in the complex plane and $a_j = a_j(z)$ $(j = 1, 2, 3, \ldots, q)$ are distinct small functions of f. Then*

$$\begin{aligned}
(q - 1)T(r, f) \le \sum_{j=1}^{q} N(r, 0; f - a_j) \\
+ q\bar{N}(r, \infty; f) + S(r, f).
\end{aligned}$$

Since for an entire function f, $\bar{N}(r, \infty; f) \equiv 0$, it is easy to note that for entire functions, Chuang's result is the second fundamental theorem for small functions as targets. After the work of C. T. Chuang, in 1986 G. Frank and G. Weissenborn [13] made a major break through by considering rational functions as targets and proved the following result.

Theorem 5. *Suppose that f is a transcendental meromorphic function in the complex plane and $a_1, a_2, \ldots, a_q$ are distinct rational functions. Then for any positive number ε we have*

$$\begin{aligned}
(q - 1 - \varepsilon)T(r, f) \le \sum_{j=1}^{q} N(r, 0; f - a_j) \\
+ N(r, \infty; f) + S(r, f).
\end{aligned}$$

In the same year 1986 N. Steinmetz [41] was able to tackle the general small functions instead of rational functions and proved the following:

Theorem 6. *Suppose that f is a transcendental meromorphic function in the complex plane and $a_1, a_2, \ldots, a_q$ are distinct small functions of f. Then for any positive number ε we have*

$$\begin{aligned}
(q - 1 - \varepsilon)T(r, f) \le \sum_{j=1}^{q} N(r, 0; f - a_j) \\
+ N(r, \infty; f) + S(r, f).
\end{aligned}$$

Proof. We write $A = (a_1, a_2, \ldots, a_q)$ and let $L(s, A)$ be the vector space generated by $a_1^{n_1} a_2^{n_2} \cdots a_q^{n_q}$, where $n_j (j = 1, 2, 3, \ldots, q)$ are nonnegative integers such that $n_1 + n_2 + \cdots + n_q = s$.

We denote by $\dim L(s, A)$, the dimension of the vector space $L(s, A)$. For a fixed s, let $\dim L(s, A) = n$ and let $b_1, b_2, \ldots, b_n$ be a basis of $L(s, A)$.

Let $\dim L(s + 1, A) = k$ and $\beta_1, \beta_2, \ldots, \beta_k$ be a basis of $L(s + 1, A)$. Then $n \le k$. We now claim that for any given $\varepsilon (> 0)$, there exists a positive integer s such that $\frac{k}{n} < 1 + \varepsilon$.

On the contrary,

$$k = \dim L(s+1, A) \geq (1+\varepsilon) \dim L(s, A) = (1+\varepsilon)n$$

holds for any natural number s. Hence

$$k \geq (1+\varepsilon)^2 \dim L(s-1, A) \geq \cdots \geq b(1+\varepsilon)^s,$$

where $b = \dim L(1, A)$ is a constant.

On the other hand,

$$k = \dim L(s+1, A) \leq \binom{q+s}{s+1} \leq cs^{q-1},$$

where c is a constant. Therefore $b(1+\varepsilon)^s \leq cs^{q-1}$ for $s = 1, 2, 3, \ldots$ and so $\log b + s\log(1+\varepsilon) \leq \log c + (q-1)\log s$, which is a contradiction. Hence for any given $\varepsilon (> 0)$, there exists an integer s such that $\frac{k}{n} < 1+\varepsilon$.

We now select s such that $\frac{k}{n} < 1+\varepsilon$. Let $P(f) = W(\beta_1, \beta_2, \ldots, \beta_k, fb_1, fb_2, \ldots, fb_n)$. Then we can write

$$P(f) = f^n \sum C_p(z) \prod_{j=0}^{n+k-1} \left(\frac{f^{(j)}}{f}\right)^{p_j},$$

where $C_p(z)$ is a small function of f. Then

$$m(r, P(f)) \leq nm(r, f) + S(r, f). \tag{6}$$

Also using a property of Wronskian determinant we have

$$P(f) = f^{n+k} W\left(\frac{\beta_1}{f}, \frac{\beta_2}{f}, \ldots, \frac{\beta_k}{f}, b_1, b_2, \ldots, b_n\right).$$

Since the poles of $P(f)$ come from the poles of $\beta_i (i = 1, 2, 3, \ldots, k)$, $b_j (j = 1, 2, \ldots, n)$ or f, we see that

$$N(r, P(f)) \leq (n+k)N(r, f) + S(r, f). \tag{7}$$

From (6) and (7) we get

$$T(r, P(f)) \leq nT(r, f) + kN(r, f) + S(r, f). \tag{8}$$

If a is a linear combination of $a_j (j = 1, 2, \ldots, q)$, then using the property of a Wronskian determinant we have

$$P(f-a) = P(f). \tag{9}$$

Also we can write

$$P(f) = f^n Q\left(\frac{f'}{f}\right), \tag{10}$$

where $Q\left(\frac{f'}{f}\right)$ is a differential polynomial in $\frac{f'}{f}$.

Let $u_j = f - a_j$ and $Q_j = Q\left(\frac{u_j'}{u_j}\right)$ for $j = 1, 2, \ldots, q$. Then from (9) and (10) we obtain $P(f) = P(u_j) = u_j^n Q_j$ for $j = 1, 2, \ldots, q$. Hence $\frac{1}{(f-a_j)^n} = \frac{Q_j}{P(f)}$ and so

$$\frac{1}{|f-a_j|} = \frac{|Q_j|^{\frac{1}{n}}}{|P(f)|^{\frac{1}{n}}}, \tag{11}$$

for $j = 1, 2, \ldots, q$.

Let $F(z) = \sum_{j=1}^{q} \frac{1}{f(z)-a_j(z)}$. Then by a result of C. T. Chuang we get

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) = m(r, F) + S(r, f). \tag{12}$$

By (11) we get

$$|F(z)| \leq \sum_{j=1}^{q} \frac{1}{|f(z)-a_j(z)|} \leq \frac{1}{|P(f)|^{\frac{1}{n}}} \sum_{j=1}^{q} |Q_j|^{\frac{1}{n}}.$$

From this and (8) we deduce that

$$\begin{aligned} m(r, F) &\leq \frac{1}{n} m\left(r, \frac{1}{P(f)}\right) + \frac{1}{n} \sum_{j=1}^{q} m(r, Q_j) + O(1) \\ &= \frac{1}{n} T(r, P(f)) - \frac{1}{n} N\left(r, \frac{1}{P(f)}\right) + S(r, f) \\ &\leq T(r, f) + \frac{k}{n} N(r, f) - \frac{1}{n} N\left(r, \frac{1}{P(f)}\right) + S(r, f). \end{aligned} \tag{13}$$

From (12) and (13) we get

$$\begin{aligned} m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) &\leq \left(1 + \frac{k}{n}\right) T(r, f) + S(r, f) \\ &\leq (2+\varepsilon) T(r, f) + S(r, f). \end{aligned}$$

Now by the first fundamental theorem we obtain

$$\begin{aligned} (q-1-\varepsilon) T(r, f) \leq \sum_{j=1}^{q} N(r, 0; f-a_j) \\ + N(r, \infty; f) + S(r, f). \end{aligned} \qquad \square$$

In 2001 H. Y. Li and Q. C. Zhang [30] improved the result of Steinmetz and proved the following theorem.

Theorem 7. *Suppose that f is a transcendental meromorphic function in the complex plane and $a_1, a_2, \ldots, a_q$ are q distinct*